

# Gram–Schmidt–Vaserstein generators for odd sized elementary groups

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**Abstract:** The Gram-Schmidt process of L.N. Vaserstein for making an elementary matrix symplectic yields a nice set of generators for the odd sized elementary group.

## 1 Introduction

The Gram–Schmidt process is a method for orthonormalising a set of vectors in an inner product space. This enables one to transform a square matrix (over the reals) by an upper triangular matrix to an orthogonal matrix.

A similar process exists in [6], due to L.N. Vaserstein, in the symplectic group  $\mathrm{Sp}_{2n}(R)$ . Here he showed that given an elementary matrix (over a commutative ring  $R$ ) of even size  $2n$  one can transform it by an elementary matrix of size  $2n - 1$  to a (elementary) symplectic matrix.

Let  $\varphi$  be an invertible alternating matrix of size  $2n$ . Then L.N. Vaserstein’s method actually permits one to transform an elementary matrix of even size  $2n$  by an elementary matrix of size  $2n - 1$  so that it is a (elementary) symplectic matrix w.r.t.  $\varphi$ .

We decided to collect all these odd sized elementary matrices which transform a given elementary matrix to a symplectic matrix w.r.t.  $\varphi$ . We call the subgroup of the elementary group  $E_{2n-1}(R)$  generated by these as the group of elementary matrices  $E_\varphi(R)$  w.r.t.  $\varphi$ .

Our main result is the following observation:

**Theorem:** For an invertible alternating matrix  $\varphi$  of size  $2n$ , the subgroup  $E_\varphi(R)$  coincides with  $E_{2n-1}(R)$ .

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This is proved by using a well-known Local-Global argument of D. Quillen in [2]; to reduce the problem to the case when  $\varphi = \psi_r$ , the standard symplectic matrix.

We have also proved a relative version of the above theorem w.r.t. an ideal of the ring  $R$ .

## 2 Preliminaries

A row  $v = (v_1, \dots, v_n) \in R^n$  is said to be *unimodular* if there are elements  $w_1, \dots, w_n$  in  $R$  such that  $v_1 w_1 + \dots + v_n w_n = 1$ .  $\text{Um}_n(R)$  will denote the set of all unimodular rows  $v \in R^n$ . Let  $I$  be an ideal in  $R$ . We denote by  $\text{Um}_n(R, I)$  the set of all unimodular rows of length  $n$  which are congruent to  $e_1 = (1, 0, \dots, 0)$  modulo  $I$ . (If  $I = R$ , then  $\text{Um}_n(R, I)$  is  $\text{Um}_n(R)$ ).

Let  $E_n(R)$  denote the subgroup of  $\text{SL}_n(R)$  consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *elementary generators*  $E_{ij}(\lambda) = I_n + e_{ij}(\lambda)$ ,  $1 \leq i \neq j \leq n$ ,  $\lambda \in R$ , where  $e_{ij}(\lambda) \in M_n(R)$  has an entry  $\lambda$  in its  $(i, j)$ -th position.

In the sequel, if  $\alpha$  denotes an  $m \times n$  matrix, then we let  $\alpha^t$  denote its *transpose* matrix. This is of course an  $n \times m$  matrix. However, we will mostly be working with square matrices, or rows and columns.

**Definition 2.1. The Relative Groups  $E_n(I)$ ,  $E_n(R, I)$ :** Let  $I$  be an ideal of  $R$ . The *relative elementary group*  $E_n(I)$  is the subgroup of  $E_n(R)$  generated as a group by the elements  $E_{ij}(x)$ ,  $x \in I$ ,  $1 \leq i \neq j \leq n$ .

The *relative elementary group*  $E_n(R, I)$  is the normal closure of  $E_n(I)$  in  $E_n(R)$ .

**Lemma 2.2.** (See [5], Corollary 1.2, Lemma 1.3) *Let  $v, w \in R^n$ , with  $n \geq 3$  and  $\langle v, w \rangle = v \cdot w^t = 0$ . Assume that  $v$  is unimodular and  $w \in I^{2n-1}(\subseteq R^{2n-1})$ . Then  $I_n + v^t w \in E_n(R, I)$ .*

**Notation 2.3.** Let  $M$  be a finitely presented  $R$ -module and  $a$  be a non-nilpotent element of  $R$ . Let  $R_a$  denote the ring  $R$  localized at the multiplicative set  $\{a^i : i \geq 0\}$  and  $M_a$  denote the  $R_a$ -module  $M$  localized at  $\{a^i : i \geq 0\}$ . Let  $\alpha(X)$  be an element of  $\text{End}(M[X])$ . The localization map  $i : M \rightarrow M_a$  induces a map  $i^* : \text{End}(M[X]) \rightarrow \text{End}(M[X]_a) = \text{End}(M_a[X])$ . We shall denote  $i^*(\alpha(X))$  by  $\alpha(X)_a$  in the sequel.

## 3 Elementary Linear Group $E_\varphi(R)$

**Definition 3.1. Alternating Matrix:** A matrix from  $M_n(R)$  is said to be *alternating* if it has the form  $\nu - \nu^t$ , where  $\nu \in M_n(R)$ . (It follows that its diagonal elements are zeros.)

**Definition 3.2.** The group of all invertible  $2n \times 2n$  matrices  $\{\alpha \in \text{GL}_{2n}(R) \mid \alpha^t \varphi \alpha = \varphi\}$ , where  $\varphi$  is an alternating matrix is called *Symplectic Group*  $\text{Sp}_\varphi(R)$  w.r.t. an invertible alternating matrix  $\varphi$ .

**Definition 3.3.** Let  $v \in R^{2n-1}$ . Let  $\varphi$  be an invertible alternating matrix of size  $2n$  of the form  $\begin{pmatrix} 0 & -c \\ c^t & \nu \end{pmatrix}$ , and  $\varphi^{-1}$  be of the form  $\begin{pmatrix} 0 & d \\ -d^t & \mu \end{pmatrix}$ , where  $c, d \in R^{2n-1}$ . In ([6], Lemma 5.4) L.N. Vaserstein considered the matrices (related to  $\varphi$  and  $v \in R^{2n-1}$ ):

$$\begin{aligned}\alpha &:= \alpha_\varphi(v) := I_{2n-1} + d^t v \nu, \\ \beta &:= \beta_\varphi(v) := I_{2n-1} + \mu v^t c.\end{aligned}$$

Note that  $\alpha_\varphi(v), \beta_\varphi(v) \in E_{2n-1}(R)$  via Lemma 2.2.

From these matrices he constructed in ([6], Lemma 5.4)

$$\begin{aligned}C_\varphi(v) &= \begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^t & \alpha \end{pmatrix} \text{ and} \\ R_\varphi(v) &= \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix}.\end{aligned}$$

(The notation  $C_\varphi(v), R_\varphi(v)$  is due to us.) In ([6], Lemma 5.4) it is mentioned that these matrices belong to  $\text{Sp}_\varphi(R)$ .

**Remark 3.4.** Using this construction, L.N. Vaserstein showed in ([6], Lemma 5.5) that given  $\varepsilon \in E_{2n}(R)$  there exists  $\rho \in E_{2n-1}(R)$ , which is a product of elements of the type  $\alpha_\varphi(v), \beta_\varphi(v)$ , such that  $(1 \perp \rho)\varepsilon \in \text{Sp}_\varphi(R)$  (We refer the reader to Lemma 4.3 for details.)

**Definition 3.5. Elementary Linear Group  $E_\varphi(R), E_\varphi(R, I)$ :** For an invertible alternating matrix  $\varphi$  of size  $2n$  and for  $v \in R^{2n-1}$ , we have  $\alpha_\varphi(v), \beta_\varphi(v) \in E_{2n-1}(R)$  (see Lemma 2.2). The subgroup of  $E_{2n-1}(R)$  generated by  $\alpha_\varphi(v), \beta_\varphi(v)$ , for all  $v \in R^{2n-1}$  is called a *linear elementary group w.r.t. to an alternating matrix  $\varphi$*  and is denoted by  $E_\varphi(R)$ .

Let  $I$  be an ideal of  $R$ . The subgroup of  $E_\varphi(R)$  generated by  $\alpha_\varphi(v), \beta_\varphi(v)$ , for all  $v \in I^{2n-1} (\subseteq R^{2n-1})$  is denoted by  $E_\varphi(I)$ . And the normal closure of  $E_\varphi(I)$  in  $E_\varphi(R)$  is denoted by  $E_\varphi(R, I)$ .

**Lemma 3.6.** (Splitting property) For two row vectors  $v, w \in R^{2n-1}$

$$\begin{aligned}\alpha_\varphi(v+w) &= \alpha_\varphi(v)\alpha_\varphi(w), \\ \beta_\varphi(v+w) &= \beta_\varphi(v)\beta_\varphi(w).\end{aligned}$$

Hence the generators of  $E_\varphi(R)$  satisfy the splitting property.

Proof: Note that

$$\begin{aligned}\alpha_\varphi(v)\alpha_\varphi(w) &= (I_{2n-1} + d^t v \nu)(I_{2n-1} + d^t w \nu) \\ &= I_{2n-1} + d^t v \nu + d^t w \nu, \quad \text{since } \nu d^t = 0 \\ &= I_{2n-1} + d^t(v+w)\nu = \alpha_\varphi(v+w),\end{aligned}$$

$$\begin{aligned}
\beta_\varphi(v)\beta_\varphi(w) &= (I_{2n-1} + \mu v^t c)(I_{2n-1} + \mu w^t c) \\
&= I_{2n-1} + \mu v^t c + \mu w^t c, \quad \text{since } c\mu = 0 \\
&= I_{2n-1} + \mu(v + w)^t c = \beta_\varphi(v + w).
\end{aligned}$$

□

The following lemma is well known.

**Lemma 3.7.** *Let  $G$  be a group, and  $a_i, b_i \in G$ , for  $i = 1, \dots, n$ . Then  $\prod_{i=1}^n a_i b_i = \prod_{i=1}^n r_i b_i r_i^{-1} \prod_{i=1}^n a_i$ , where  $r_i = \prod_{j=1}^i a_j$ .* □

**Lemma 3.8.** *Let  $R[X]$  be the polynomial ring and  $(X)$  be the ideal generated by  $X$ . Let  $\varphi$  be an invertible alternating matrix of size  $2n$ . Then  $E_\varphi(R[X], (X)) = E_\varphi(R[X]) \cap \text{GL}_{2n-1}(R[X], (X))$ .*

Proof: It is easy to see that  $E_\varphi(R[X], (X)) \subseteq E_\varphi(R[X]) \cap \text{GL}_{2n-1}(R[X], (X))$ . The other way inclusion follows from Lemma 3.6 and Lemma 3.7. □

**Lemma 3.9.** *If  $\varphi = \psi_n$ , then  $E_\varphi(R) = E_{2n-1}(R)$ .*

Proof:  $E_\varphi(R) \subseteq E_{2n-1}(R)$  (see Lemma 2.2). Let  $e_i$  denote a row vector of length  $2n - 1$  which has 1 at the  $i^{\text{th}}$  place and zeros elsewhere. When  $\varphi = \psi_n$ , then for an element  $a \in R$  we have

$$\begin{aligned}
E_{12}(a) &= \alpha_{\psi_n}(a.e_3), \\
E_{1j}(a) &= \begin{cases} \alpha_{\psi_n}(a.e_{j+1}) & \text{if } j \geq 3 \text{ and } j \text{ is even,} \\ \alpha_{\psi_n}(-a.e_{j-1}) & \text{if } j \geq 3 \text{ and } j \text{ is odd,} \end{cases} \\
E_{21}(a) &= \beta_{\psi_n}(a.e_3), \\
E_{i1}(a) &= \begin{cases} \beta_{\psi_n}(a.e_{i+1}) & \text{if } i \geq 3 \text{ and } i \text{ is even,} \\ \beta_{\psi_n}(-a.e_{i-1}) & \text{if } i \geq 3 \text{ and } i \text{ is odd,} \end{cases}
\end{aligned}$$

and hence  $E_{2n-1}(R) \subseteq E_\varphi(R)$ . □

**Lemma 3.10.** *Let  $\varphi$  and  $\varphi^*$  be two invertible alternating matrices and*

$$\varphi = (1 \perp \varepsilon)^t \varphi^* (1 \perp \varepsilon),$$

*for some  $\varepsilon \in E_{2n-1}(R)$ . Then  $E_\varphi(R) = \varepsilon^{-1} E_{\varphi^*}(R) \varepsilon$ .*

Proof: Note that when  $\varphi^*$  is of the form  $\begin{pmatrix} 0 & -c \\ c^t & \nu \end{pmatrix}$ , and  $\varphi^{*-1}$  is of the form  $\begin{pmatrix} 0 & d \\ -d^t & \mu \end{pmatrix}$ , where  $c, d \in R^{2n-1}$ , then

$$\varphi = \begin{pmatrix} 0 & -c\varepsilon \\ \varepsilon^t c^t & \varepsilon^t \nu \varepsilon \end{pmatrix} \text{ and } \varphi^{-1} = \begin{pmatrix} 0 & d(\varepsilon^t)^{-1} \\ -\varepsilon^{-1} d^t & \varepsilon^{-1} \mu (\varepsilon^t)^{-1} \end{pmatrix}.$$

Using the definition of  $\alpha_\varphi$  and  $\beta_\varphi$  we get

$$\begin{aligned}
\alpha_\varphi(v) &= Id + (\varepsilon^{-1} d^t) v (\varepsilon^t \nu \varepsilon) = \varepsilon^{-1} (Id + d^t (v \varepsilon^t) \nu) \varepsilon = \varepsilon^{-1} \alpha_{\varphi^*}(v \varepsilon^t) \varepsilon, \\
\beta_\varphi(v) &= Id + (\varepsilon^{-1} \mu (\varepsilon^t)^{-1}) v^t (c \varepsilon) = \varepsilon^{-1} (Id + \mu ((\varepsilon^t)^{-1} v^t) c) \varepsilon = \varepsilon^{-1} \beta_{\varphi^*}(v \varepsilon^{-1}) \varepsilon,
\end{aligned}$$

and hence the equality follows. □

**Lemma 3.11.** *Let  $\varphi = (1 \perp \varepsilon)^t \psi_n(1 \perp \varepsilon)$ , for some  $\varepsilon \in E_{2n-1}(R)$ . Then  $E_\varphi(R) = E_{2n-1}(R)$ .*

Proof: Follows from the above two lemmas.  $\square$

## 4 Dilation and LG Principle for $E_{\varphi \otimes R[X]}(R[X])$

Here we establish dilation principle and Local-Global principle for  $E_{\varphi \otimes R[X]}(R[X])$ .

**Lemma 4.1.** (Dilation principle) *Let  $\varphi$  be an alternating matrix of Pfaffian 1 of size  $2n$ , with  $n \geq 2$ . Let  $a \in R$  be a non-nilpotent element, and let  $\varphi = (1 \perp \varepsilon)^t \psi_n(1 \perp \varepsilon)$ , for some  $\varepsilon \in E_{2n-1}(R_a)$  over the ring  $R_a$ . Let  $\theta(X) \in E_{\varphi \otimes R_a[X]}(R_a[X])$ , with  $\theta(0) = Id$ . Then there exists  $\theta^*(X) \in E_{\varphi \otimes R[X]}(R[X])$  such that  $\theta^*(X)$  localises to  $\theta(bX)$ , for some  $b \in (a^d)$ ,  $d \gg 0$ , and  $\theta^*(0) = Id$ .*

Proof: We are given that  $\theta(X) \in E_{\varphi \otimes R_a[X]}(R_a[X])$ , where  $\varphi = (1 \perp \varepsilon)^t \psi_n(1 \perp \varepsilon)$ , for some  $\varepsilon \in E_{2n-1}(R_a)$  over the ring  $R_a$ . Therefore by Lemma 3.10 we have  $\theta(X) = \varepsilon^{-1} \eta(X) \varepsilon$ , for some  $\eta(X) \in E_{2n-1}(R_a[X])$ . Since  $\eta(0) = Id$ , we can write  $\eta(X) = \prod \gamma_l E_{i_l j_l}(X f_l(X)) \gamma_l^{-1}$ , where  $\gamma_l \in E_{2n-1}(R_a)$ , and  $f_l(X) \in R_a[X]$  (see Lemma 3.7). Using commutator identities for the generators of the elementary linear group we get  $\eta(Y^r X) = \prod E_{i_k j_k}(Y h_k(X, Y)/a^s)$ , for large integer  $r$ . Here  $h_k(X, Y) \in R[X, Y]$  and either  $i_k = 1$  or  $j_k = 1$ . Using equalities appearing in the Lemma 3.9 it is clear that  $\eta(Y^r X)$  is product of the elements of the form  $\alpha_{\psi_n}((Y h_k(X, Y)/a^s).e_i)$  or  $\beta_{\psi_n}((Y h_k(X, Y)/a^s).e_j)$ , where  $2 \leq i, j \leq 2n-1$ .

Note that  $\alpha_{\psi_n}((Y h_k(X, Y)/a^s).e_i) = \varepsilon \alpha_\varphi((Y h_k(X, Y)/a^s).e_i(\varepsilon^t)^{-1}) \varepsilon^{-1}$  and  $\beta_{\psi_n}((Y h_k(X, Y)/a^s).e_j) = \varepsilon \beta_\varphi((Y h_k(X, Y)/a^s).e_j \varepsilon) \varepsilon^{-1}$ . Therefore  $\theta(Y^r X)$  is product of elements of the form  $\alpha_\varphi((Y h_k(X, Y)/a^s).e_i(\varepsilon^t)^{-1})$  or  $\beta_\varphi((Y h_k(X, Y)/a^s).e_j \varepsilon)$ . Let  $t$  be the maximum power of  $a$  appearing in the denominators of  $\varepsilon$  and  $(\varepsilon^t)^{-1}$ . Set  $d = s+t$ . Define  $\theta^*(X, Y)$  as product of elements of the form  $\alpha_\varphi(Y h_k(X, a^d Y).a^t e_i(\varepsilon^t)^{-1})$  and  $\beta_\varphi(Y h_k(X, a^d Y).a^t e_j \varepsilon)$ . Note that  $\theta^*(X, Y) \in E_{\varphi \otimes R[X, Y]}(R[X, Y])$ . We obtain  $\theta^*(X)$  substituting  $Y = 1$  in  $\theta^*(X, Y)$ . Clearly  $\theta^*(X)$  localises to  $\theta(bX)$  for some  $b \in (a^d)$ , and  $\theta^*(0) = Id$ .  $\square$

**Remark 4.2.** *Let  $(R, \mathfrak{m})$  be a local ring and  $\varphi$  be an alternating matrix of Pfaffian 1 over  $R$  of size  $2n$ . Then  $\varphi = \varepsilon^t \psi_n \varepsilon$ , for some  $\varepsilon \in E_{2n}(R)$ .*

We recollect an observation of Rao-Swan stated in the introduction of [4]. We make a contextual observation which the proof shows and include it for completeness.

**Lemma 4.3.** (Rao-Swan) *Let  $n \geq 2$  and  $\varepsilon \in E_{2n}(R)$ . Then there exists  $\rho \in E_{\psi_n}(R) \subseteq E_{2n-1}(R)$  such that  $(1 \perp \rho)\varepsilon \in \text{ESp}_{2n}(R)$ .*

Proof: Let  $\varepsilon = \varepsilon_r \dots \varepsilon_1$ , where each  $\varepsilon_i$  is of the form  $\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix}$ , where  $v = (a_1, \dots, a_{2n-1}) \in R^{2n-1}$  (see [6], Lemma 2.7). We prove the result using induction on  $r$ . It is clear when  $r = 0$ . Let  $r \geq 1$ . Let us assume the result is true for  $r-1$ , i.e., when  $\varepsilon = \varepsilon_{r-1} \dots \varepsilon_1$ , then there exists a  $\delta \in E_{2n-1}(R)$  such that  $(1 \perp \delta)\varepsilon \in \text{ESp}_{2n}(R)$ .

We will prove the result when number of generators of  $\varepsilon$  is  $r$ . We have

$$\begin{aligned} C_{\psi_n}(v) &= \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{i1}(a_{i-1}), \\ R_{\psi_n}(v) &= \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{1i}(a_{i-1}). \end{aligned}$$

Note that  $\alpha = \alpha_{\psi_n}(v), \beta = \beta_{\psi_n}(v) \in E_{2n-1}(R)$ . Let us set  $\gamma$  equal to either  $\alpha$  or  $\beta$  depending on the form of  $\varepsilon_1$ . Now,  $\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1 \in \text{ESp}_{2n}(R)$ , and each  $\eta_i = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_i \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$  is of the form  $\begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix}$  or  $\begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix}$ . Now we have

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \eta_r \dots \eta_2 \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1.$$

By induction hypothesis  $(1 \perp \delta) \eta_r \dots \eta_2 \in \text{ESp}_{2n}(R)$ , for some  $\delta \in E_{2n-1}(R)$ . Hence  $(1 \perp \rho) \varepsilon \in \text{ESp}_{2n}(R)$ , where  $\rho = \delta^{-1} \gamma \in E_{2n-1}(R)$ .  $\square$

**Corollary 4.4.** (Rao-Swan) *For  $n \geq 2$  and  $\varepsilon \in E_{2n}(R)$ , we have an  $\varepsilon_0 \in E_{\psi_n}(R) \subseteq E_{2n-1}(R)$  such that  $\varepsilon^t \psi_n \varepsilon = (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0)$ .*

Proof: Using Lemma 4.3 we get  $\varepsilon_0 \in E_{\psi_n}(R) \subseteq E_{2n-1}(R)$  such that  $(1 \perp \varepsilon_0) \varepsilon^{-1} \in \text{ESp}_{2n}(R)$ , and hence  $\varepsilon^{-1t} (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0) \varepsilon^{-1} = \psi_n$ .

Therefore we have

$$\begin{aligned} \varepsilon^t \psi_n \varepsilon &= \varepsilon^t \{ \varepsilon^{-1t} (1 \perp \varepsilon_0)^t \} \psi_n \{ (1 \perp \varepsilon_0) \varepsilon^{-1} \} \varepsilon \\ &= (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0). \end{aligned}$$

$\square$

**Corollary 4.5.** *Let  $(R, \mathfrak{m})$  be a local ring and  $\varphi$  be an alternating matrix of Pfaffian 1 over  $R$  of size  $2n$  with  $n \geq 2$ . Then the groups  $E_{2n-1}(R)$  and  $E_\varphi(R)$  are equal.*

Proof: Over the local ring  $(R, \mathfrak{m})$  we have  $\varphi = (1 \perp \varepsilon_0)^t \psi_n (1 \perp \varepsilon_0)$ , for some  $\varepsilon_0 \in E_{2n-1}(R)$ . This follows by Remark 4.2 and Corollary 4.4. Using Lemma 3.10 we get  $E_\varphi(R) = \varepsilon_0^{-1} E_{\psi_n}(R) \varepsilon_0$ . Using Lemma 3.9 we get  $E_{\psi_n}(R) = E_{2n-1}(R)$ , and hence the equality follows.  $\square$

Using dilation principle we prove the following variant of D. Quillen's Local-Global principle (see [2]). The argument is standard. We include the proof for completeness. Interested reader may refer to [1] for a survey about the Local-Global principle.

**Theorem 4.6.** (Local-Global principle) *Let  $\varphi$  be an alternating matrix of Pfaffian 1 of size  $2n$ , with  $n \geq 2$ . Let  $\theta(X) \in \text{SL}_{2n-1}(R[X])$ , with  $\theta(0) = \text{Id}$ . If  $\theta(X)_\mathfrak{m} \in E_{\varphi \otimes R_\mathfrak{m}[X]}(R_\mathfrak{m}[X])$ , for all maximal ideal  $\mathfrak{m}$  of  $R$ , then  $\theta(X) \in E_{\varphi \otimes R[X]}(R[X])$ .*

Proof: For each maximal ideal  $\mathfrak{m}$  of  $R$  one can suitably choose an element  $a_{\mathfrak{m}}$  from  $R \setminus \mathfrak{m}$  such that  $\theta(X)_{a_{\mathfrak{m}}} \in E_{\varphi \otimes R_{a_{\mathfrak{m}}}[X]}(R_{a_{\mathfrak{m}}}[X])$  and also one has  $\varphi = (1 \perp \varepsilon)^t \psi_n (1 \perp \varepsilon)$ , for some  $\varepsilon \in E_{2n-1}(R_{a_{\mathfrak{m}}})$ . Define  $\gamma(X, Y) = \theta(X + Y)_{a_{\mathfrak{m}}} \theta(Y)_{a_{\mathfrak{m}}}^{-1}$ . It is clear that

$$\gamma(X, Y) \in E_{\varphi \otimes R_{a_{\mathfrak{m}}}[X, Y]}(R_{a_{\mathfrak{m}}}[X, Y])$$

and  $\gamma(0, Y) = Id$ . Therefore  $\gamma(b_{\mathfrak{m}}X, Y) \in E_{\varphi \otimes R[X, Y]}(R[X, Y])$ , where  $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^d)$  for  $d \gg 0$  (see Lemma 4.1). Note that the ideal generated by  $a_{\mathfrak{m}}^d$ 's is the whole ring  $R$ . Therefore,  $c_1 a_{\mathfrak{m}_1}^d + \dots + c_k a_{\mathfrak{m}_k}^d = 1$ , for some  $c_i \in R$ . Let  $b_{m_i} = c_i a_{m_i}^d \in (a_{m_i}^d)$ . It is easy to see that  $\theta(X) = \prod_{i=1}^{k-1} \gamma(b_{m_i}X, T_i) \gamma(b_{m_k}X, 0)$ , where  $T_i = b_{m_{i+1}}X + \dots + b_{m_k}X$ . Each term in the right hand side of this expression belongs to  $E_{\varphi \otimes R[X]}(R[X])$ , and hence  $\theta(X) \in E_{\varphi \otimes R[X]}(R[X])$ .  $\square$

We recall Swan-Weibel's trick to establish the Local-Global principle in the graded case.

**Theorem 4.7.** (Graded case of Local-Global principle) *Let  $\varphi$  be an alternating matrix of Pfaffian 1 of size  $2n$ , with  $n \geq 2$ . Let  $\theta(X_1, \dots, X_t) \in \mathrm{SL}_{2n-1}(R[X_1, \dots, X_t])$ , with  $\theta(0, \dots, 0) = Id$ . If  $\theta(X_1, \dots, X_t)_{\mathfrak{m}} \in E_{\varphi \otimes R_{\mathfrak{m}}[X_1, \dots, X_t]}(R_{\mathfrak{m}}[X_1, \dots, X_t])$ , for all maximal ideal  $\mathfrak{m}$  of  $R$ , then  $\theta(X_1, \dots, X_t) \in E_{\varphi \otimes R[X_1, \dots, X_t]}(R[X_1, \dots, X_t])$ .*

Proof: Let us denote  $S = R[X_1, \dots, X_t]$ . Note that  $S$  is a graded ring with the grading  $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$ , and  $S_0 = R$ . Consider the ring homomorphism  $f : S \rightarrow S[T]$  given by  $f(a_0 + a_1 + a_2 + \dots) = a_0 + a_1T + a_2T^2 + \dots$ , where each  $a_i$  is a homogeneous component belongs to  $S_i$ . Let us denote  $\theta(X_1, \dots, X_t) = (\theta_{ij})$ , where  $\theta_{ij} \in S$ . We set  $\tilde{\theta}(T) = (f(\theta_{ij}))$ . Note that  $\tilde{\theta}(1) = (\theta_{ij})$ , and  $\tilde{\theta}(0) = \theta(0, \dots, 0) = Id$ .

Let  $\mathfrak{m}_0$  be a maximal ideal of  $R$  and let  $M_0 = R \setminus \mathfrak{m}_0$ . Since  $(\theta_{ij})_{M_0} \in E_{\varphi}(S_{M_0})$ , we have  $\tilde{\theta}(T)_{M_0} \in E_{\varphi \otimes S_{M_0}[T]}(S_{M_0}[T])$ . Therefore, there is a  $s_{m_0} \in M_0$  such that  $\tilde{\theta}(T)_{s_{m_0}} \in E_{\varphi \otimes S_{s_{m_0}}[T]}(S_{s_{m_0}}[T])$ . If  $\mathfrak{m}$  is a maximal ideal of  $S$  then  $s_{m_0} \notin \mathfrak{m}$  for some  $\mathfrak{m}_0$ . Therefore,  $\tilde{\theta}(T)_{\mathfrak{m}} \in E_{\varphi \otimes S_{\mathfrak{m}}[T]}(S_{\mathfrak{m}}[T])$ , for all maximal ideals  $\mathfrak{m}$  of  $S$ . Moreover, the ideal generated by all  $s_{m_0}$ , for all maximal ideals  $\mathfrak{m}_0$  of  $R$ , is the whole ring  $R$ . Hence,  $\tilde{\theta}(T) \in E_{\varphi}(S[T])$ . Now substituting  $T = 1$  we get  $\tilde{\theta}(1) = (\theta_{ij}) = \theta(X_1, \dots, X_t) \in E_{\varphi \otimes R[X_1, \dots, X_t]}(R[X_1, \dots, X_t])$ .  $\square$

## 5 Equality of $E_{2n-1}(R)$ and $E_{\varphi}(R)$

We are now ready to prove the main theorem of this note.

**Theorem 5.1.** *Let  $\varphi$  be an alternating matrix of Pfaffian 1 of size  $2n$ , with  $n \geq 2$ . Then the groups  $E_{2n-1}(R)$  and  $E_{\varphi}(R)$  are equal.*

Proof: By Lemma 2.2, it follows that  $E_{\varphi}(R) \subseteq E_{2n-1}(R)$ . Let  $\lambda \in E_{2n-1}(R)$ . Then there exists  $\lambda(X) \in E_{2n-1}(R[X])$  such that  $\lambda(1) = \lambda$  and  $\lambda(0) = Id$ .

Let  $\mathfrak{m}$  be a maximal ideal of  $R$  and  $R_{\mathfrak{m}}$  be the local ring at  $\mathfrak{m}$ . Over the local ring  $R_{\mathfrak{m}}$ , we have  $\varphi = (1 \perp \varepsilon)^t \psi_n (1 \perp \varepsilon)$ , for some  $\varepsilon \in E_{2n-1}(R_{\mathfrak{m}})$ . Therefore, for each

maximal ideal  $\mathfrak{m}$  of  $R$ , we have  $\lambda(X)_{\mathfrak{m}} \in E_{2n-1}(R_{\mathfrak{m}}[X]) = E_{\varphi \otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X])$  (see Corollary 4.5). By Theorem 4.6 it follows that  $\lambda(X) \in E_{\varphi \otimes R[X]}(R[X])$ . Put  $X = 1$  to get  $\lambda = \lambda(1) \in E_{\varphi}(R)$ , i.e.,  $E_{2n-1}(R) \subseteq E_{\varphi}(R)$ . Hence the equality follows.  $\square$

Using the ideas of the proof of ([3], Lemma 3.1) we can prove a relative version of the above theorem w.r.t. an ideal.

**Theorem 5.2.** *Let  $I$  be an ideal of the ring  $R$ . Let  $\varphi$  be an alternating matrix of Pfaffian 1 of size  $2n$ , with  $n \geq 2$ . Then the groups  $E_{2n-1}(R, I)$  and  $E_{\varphi}(R, I)$  are equal.*

Proof: It is clear that  $E_{\varphi}(R, I) \subseteq E_{2n-1}(R, I)$  (see Lemma 2.2). Let  $\lambda \in E_{2n-1}(R, I)$  and  $\lambda = \prod_{k=1}^t \gamma_k E_{i_k j_k}(a_k) \gamma_k^{-1}$ , where  $\gamma_k \in E_{2n-1}(R)$  and  $a_k \in I$ . We define  $\Lambda(X_1, \dots, X_t) = \prod_{k=1}^t \gamma_k E_{i_k j_k}(X_k) \gamma_k^{-1}$ , which is in  $E_{2n-1}(R[X_1, \dots, X_t], (X_1, \dots, X_t))$ . By Corollary 4.5 for all maximal ideals  $\mathfrak{m}$  of  $R$  we have

$$E_{2n-1}(R_{\mathfrak{m}}[X_1, \dots, X_t]) = E_{\varphi \otimes R_{\mathfrak{m}}[X_1, \dots, X_t]}(R_{\mathfrak{m}}[X_1, \dots, X_t]).$$

Therefore,  $\Lambda(X_1, \dots, X_t)_{\mathfrak{m}} \in E_{\varphi \otimes R_{\mathfrak{m}}[X_1, \dots, X_t]}(R_{\mathfrak{m}}[X_1, \dots, X_t])$ , for all maximal ideals  $\mathfrak{m}$  of  $R$ . Hence,  $\Lambda(X_1, \dots, X_t) \in E_{\varphi \otimes R[X_1, \dots, X_t]}(R[X_1, \dots, X_t])$  by Theorem 4.7. Also,  $\Lambda(X_1, \dots, X_t) \in \text{SL}_{2n-1}(R[X_1, \dots, X_t], (X_1, \dots, X_t))$  and hence  $\Lambda(X_1, \dots, X_t)$  is in the relative group  $E_{\varphi \otimes R[X_1, \dots, X_t]}(R[X_1, \dots, X_t], (X_1, \dots, X_t))$  (see Lemma 3.8). Substituting  $(X_1, \dots, X_t) = (a_1, \dots, a_t)$  we get  $\lambda \in E_{\varphi}(R, I)$ .  $\square$

**Remark 5.3.** The condition that the alternating matrices, in this article, are of Pfaffian one can be extended to all invertible alternating matrices by observing that an invertible alternating matrix over a local ring which is congruent to  $(u \psi_1 \perp \psi_{n-1}) \pmod{I}$ , where  $u = \text{Pfaffian } \varphi$ , is of the form  $(1 \perp E)^t (u \psi_1 \perp \psi_{n-1}) (1 \perp E)$ , for some relative elementary matrix  $E$ .

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